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The Threshold Expansion of the 2-loop Sunrise Selfmass Master Amplitudes.

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Abstract

The threshold behavior of the master amplitudes for two loop sunrise self-mass graph is studied by solving the system of differential equations, which they satisfy. The expansion at the threshold of the master amplitudes is obtained analytically for arbitrary masses.

PACS 11.10.-z Field theory
PACS 11.10.Kk Field theories in dimensions other than four
PACS 11.15.Bt General properties of perturbation theory
PACS 12.20.Ds Specific calculations
PACS 12.38.Bx Perturbative calculations

1 Introduction.

The sunrise graph (also known as sunset or London transport diagram) appears naturally, as a consequence of tensorial reduction, in several higher order calculations in gauge theories. Due to the presence of heavy quarks, vector bosons and Higgs particles all the internal lines may carry different masses, so that sunrise amplitudes depend in general on three different internal masses m_i , $i = 1, 2, 3$, besides the external scalar variable p^2 , if p_μ is the external momentum (in n -dimensional Euclidean space).

For a proper understanding of their analytical behaviour, as well as for a check of the numerical calculations, it is convenient to know the amplitudes off-shell, and also around some particular values of p^2 , such as $p^2 = 0$, $p^2 = \infty$, $p^2 = -(m_1 + m_2 - m_3)^2$ (one of the pseudothresholds) and $p^2 = -(m_1 + m_2 + m_3)^2$ (the threshold).

This paper is devoted to the analytic evaluation of the coefficients of the expansion of the sunrise amplitudes in p^2 , at the threshold value $-p^2 = (m_1 + m_2 + m_3)^2 \equiv s_0$. The approach relies on the exploitation of the information contained in the linear system of first order differential equations in p^2 , which is known to be satisfied by the sunrise amplitudes themselves [1]. It is to be noted that all the above points ($p^2 = 0$, ∞ , threshold and pseudothresholds) correspond to the Fuchsian points of the differential equations, which therefore emerge as a natural tool for their discussion.

The analytic properties of Feynman diagrams at threshold and pseudothresholds are well known, see for example [2]. The sunrise diagram, with different masses, has been investigated in [3], while in [4] the values of the amplitudes at threshold and pseudothreshold were obtained. With the method established in [5] and [6], further, the expansion around threshold was obtained in [7], even if a complete analytical result was given there only for the case of equal masses. With the configuration space technique the expansion around threshold was investigated also in [8]. The expansion around one of the pseudothresholds (the others are straightforwardly given by a cyclic permutation of the masses) was obtained in analytical form in [9].

The threshold is a Fuchsian point of the system of equations obtained in [1]. In this case it is known [10] that the master amplitudes can be expanded around that point as a combination of several terms, each equal to a leading power x^{α_i} times a power series in x , with $x = (p^2 + (m_1 + m_2 + m_3)^2)$, where the exponents α_i are in general real numbers.

When the expansions are inserted in the differential equations, the equations become a set of algebraic equations in the α_i and in the coefficients of the expansions; the obtained algebraic equations can then be solved recursively, for arbitrary value of the dimension

n , once the initial conditions (*i.e.* in the case considered here the values of the sunrise amplitudes at the threshold) are given. As those initial values are in turn functions of the masses, we find in our approach that the initial values themselves satisfy a system of linear differential equations in the masses. We expand in the dimension n around $n = 4$ the equations in the masses and solve them explicitly up to the finite part in $(n - 4)$. This result presented in the next section for $n = 4$ and $p^2 = -(m_1 + m_2 + m_3)^2$ is in agreement with the literature [4].

Once the initial conditions are obtained, we look at the recursive solution of the algebraic equations for the coefficients of the expansion in $(p^2 + s_0)$ (our results are given in the Section 3). It turns out that the formula, expressing the second order coefficient of the $(p^2 + s_0)$ in the first master integral expansion in terms of the zeroth order values of the other master integrals, involves the coefficient $1/(n - 4)$. Therefore the finite part at $(n - 4)$ of the second order term in the $(p^2 + s_0)$ expansion involve the first order terms in $(n - 4)$ of the zeroth order terms in $(p^2 + s_0)$. We evaluate those first order terms in $(n - 4)$ solving a differential equation, which they satisfy, obtained also in Section 3. Fortunately, the formulae expressing the higher order coefficients of the $(p^2 + s_0)$ expansion involve coefficients like $1/(n - 5)$, $1/(n - 6)$ etc., which are finite at $n = 4$, and can be used without further problems for evaluating those higher order terms. Our results agree in the equal mass case with those obtained in [7] (we take a numerical constant from the comparison), while for not equal masses the result is given in an analytical form for the first time.

2 The threshold values of the master amplitudes

It is known that the two-loop sunrise self-mass graph with arbitrary masses m_1, m_2, m_3 has four independent master amplitudes [12], which will be referred to, as in [1], by

$$F_\alpha(n, m_1^2, m_2^2, m_3^2, p^2), \quad \alpha = 0, 1, 2, 3, \quad (1)$$

where n is the continuous number of dimensions, $m_i, i = 1, 2, 3$ the three masses and p_μ the external n -momentum. $F_0(n, m_1^2, m_2^2, m_3^2, p^2)$ is the scalar amplitude

$$F_0(n, m_1^2, m_2^2, m_3^2, p^2) = \int \frac{d^n k_1}{(2\pi)^{n-2}} \int \frac{d^n k_2}{(2\pi)^{n-2}} \frac{1}{(k_1^2 + m_1^2)(k_2^2 + m_2^2)((p - k_1 - k_2)^2 + m_3^2)}, \quad (2)$$

while for $i = 1, 2, 3$

$$F_i(n, m_1^2, m_2^2, m_3^2, p^2) = -\frac{\partial}{\partial m_i^2} F_0(n, m_1^2, m_2^2, m_3^2, p^2). \quad (3)$$

The values of the master integrals at the threshold can be obtained in a way similar to the way followed for the values at the pseudothreshold [9] and we will discuss here mostly those points of the derivation which are different from the pseudothreshold case. If the values at the threshold are written as

$$\mathcal{G}_\alpha(n, m_1, m_2, m_3) \equiv F_\alpha \left(n, m_1^2, m_2^2, m_3^2, p^2 = -(m_1 + m_2 + m_3)^2 \right) \quad , \quad \alpha = 0, 1, 2, 3 \quad (4)$$

we find, expanding around $n = 4$,

$$\begin{aligned} \mathcal{G}_\alpha(n, m_1, m_2, m_3) = & C^2(n) \left[\frac{1}{(n-4)^2} \mathcal{G}_\alpha^{(-2)}(m_1, m_2, m_3) + \frac{1}{(n-4)} \mathcal{G}_\alpha^{(-1)}(m_1, m_2, m_3) \right. \\ & \left. + \mathcal{G}_\alpha^{(0)}(m_1, m_2, m_3) + O(n-4) \right] \quad \alpha = 0, 1, 2, 3 . \end{aligned} \quad (5)$$

The coefficient $C(n)$ is a function of n only and at $n = 4$ has the following expansion

$$\begin{aligned} C(n) = & (2\sqrt{\pi})^{(4-n)} \Gamma \left(3 - \frac{n}{2} \right) \\ = & 1 - (n-4) \left[\log(2\sqrt{\pi}) - \frac{1}{2} \gamma_E \right] \\ + & (n-4)^2 \frac{1}{2} \left[\log^2(2\sqrt{\pi}) + \frac{1}{4} \left(\frac{\pi^2}{6} + \gamma_E^2 \right) - \gamma_E \log(2\sqrt{\pi}) \right] \\ + & \mathcal{O} \left((n-4)^3 \right) , \end{aligned} \quad (6)$$

where γ_E is the Euler-Mascheroni constant.

From [1], where the singular parts in $(n-4)$ of $F_\alpha(n, m_1^2, m_2^2, m_3^2, p^2)$ are given for arbitrary values of p^2 , we have

$$\begin{aligned} \mathcal{G}_0^{(-2)}(m_1, m_2, m_3) &= -\frac{1}{8}(m_1^2 + m_2^2 + m_3^2) , \\ \mathcal{G}_0^{(-1)}(m_1, m_2, m_3) &= -\frac{1}{32}(m_1 + m_2 + m_3)^2 + \frac{3}{16}(m_1^2 + m_2^2 + m_3^2) \\ &\quad - \frac{1}{8} \left[m_1^2 \log(m_1^2) + m_2^2 \log(m_2^2) + m_3^2 \log(m_3^2) \right] , \\ \mathcal{G}_i^{(-2)}(m_1, m_2, m_3) &= \frac{1}{8} \quad , \quad \mathcal{G}_i^{(-1)}(m_1, m_2, m_3) = -\frac{1}{16} + \frac{1}{8} \log(m_i^2) \quad , \quad i = 1, 2, 3. \end{aligned} \quad (7)$$

To obtain the other coefficients of Eq.(5), as in [9], our starting point is the system of four linear differential equations in p^2 , satisfied by the four master amplitudes

$F_\alpha(n, m_1^2, m_2^2, m_3^2, p^2)$, $\alpha = 0, 1, 2, 3$, which are not rewritten here for short. By imposing the condition Eq.(4) to p^2 the system becomes equivalent to a system of three linear differential equations in m_3 , which in turn can be written as a single third order differential equation in m_3 for the function $\mathcal{G}_0^{(0)}(m_1, m_2, m_3)$. We do not repeat here the details of the derivation, which is very similar to the derivation leading to the analogous Eq.(17) of [9] (with the important difference of the substitution $m_3 \rightarrow -m_3$). Its solution can also be obtained again in an analogous way, for positive values of m_1, m_2 and m_3 ; it contains three unknown constants of integration (indeed functions of the masses m_1 and m_2) $\mathcal{C}_i(m_1, m_2)$, $i = 1, 2, 3$, to be fixed by the initial conditions.

The initial conditions have to be imposed however in a way different with respect to the pseudothreshold. The point $p^2 = 0$, which is known [1] and was used to impose initial conditions at the pseudothreshold [9], corresponds in the case of the threshold to the value $m_3 = -(m_1 + m_2)$, which is outside the validity of the present solution for $\mathcal{G}_0^{(0)}(m_1, m_2, m_3)$, where positive values of the masses are assumed. However we know already the value of the function at the pseudothreshold [4, 9] and for $m_3 = 0$ the threshold and pseudothreshold are identical. That allows us to choose that point for the initial conditions. Imposing that, at $m_3 = 0$, $\mathcal{G}_0^{(0)}(m_1, m_2, m_3)$, given by the Eq.(25) of [9], and its first derivative with respect to m_3 are equal to the function $\mathcal{G}_0^{(0)}(m_1, m_2, m_3)$ and to its first m_3 derivative respectively, one gets two relations between the three functions $\mathcal{C}_i(m_1, m_2)$, $i = 1, 2, 3$. The second derivative at that point is infinite and does not give an additional relation. After eliminating the two coefficients $\mathcal{C}_i(m_1, m_2)$, $i = 2, 3$ one gets $\mathcal{G}_0^{(0)}(m_1, m_2, m_3)$ as a function of $\mathcal{C}_1(m_1, m_2)$ only. The remaining constant can be fixed by the relations obtained by the permutation of the masses as we know that the $\mathcal{G}_0^{(0)}(m_1, m_2, m_3)$ is symmetric under that exchange. Once all the constants are fixed the solution reads:

$$\begin{aligned} \mathcal{G}_0^{(0)}(m_1, m_2, m_3) = & \\ & -\frac{1}{8(m_1 + m_2 + m_3)^2} \left[m_1^3(m_1 + 2m_2)\mathcal{L}_t(m_1, m_2, m_3) + m_2^3(2m_1 + m_2)\mathcal{L}_t(m_2, m_1, m_3) \right. \\ & \quad \left. - 2\pi(m_1m_2 + m_1m_3 + m_2m_3)\sqrt{m_1m_2m_3(m_1 + m_2 + m_3)} \right] \\ & + \frac{1}{4(m_1 + m_2 + m_3)} \left[(m_1 + m_2) \left(m_1^2\mathcal{L}_t(m_1, m_2, m_3) + m_2^2\mathcal{L}_t(m_2, m_1, m_3) \right) \right. \\ & \quad \left. + \frac{1}{2}(m_1^2 + m_2^2 + m_1m_2) \left(m_1 \log\left(\frac{m_3}{m_1}\right) + m_2 \log\left(\frac{m_3}{m_2}\right) \right) \right] \end{aligned}$$

$$\begin{aligned}
& +\frac{1}{8}(m_3^2 - m_1^2 - m_2^2) (\mathcal{L}_t(m_1, m_2, m_3) + \mathcal{L}_t(m_2, m_1, m_3)) \\
& -\frac{1}{32} \left[m_1^2 \log^2(m_1^2) + m_2^2 \log^2(m_2^2) + m_3^2 \log^2(m_3^2) + (m_1^2 + m_2^2 - m_3^2) \log(m_1^2) \log(m_2^2) \right. \\
& + (m_1^2 - m_2^2 + m_3^2) \log(m_1^2) \log(m_3^2) + (-m_1^2 + m_2^2 + m_3^2) \log(m_2^2) \log(m_3^2) \\
& - m_1(7m_1 - 2m_3) \log(m_1^2) - m_2(7m_2 - 2m_3) \log(m_2^2) \\
& \left. + (2m_1^2 + 2m_1m_2 + 2m_2^2 - 5m_3^2) \log(m_3^2) \right] - \frac{\pi}{4} \sqrt{m_1m_2m_3(m_1 + m_2 + m_3)} \\
& - \frac{11}{128}(m_1^2 + m_2^2 + m_3^2) + \frac{13}{64}(m_1m_2 + m_1m_3 + m_2m_3)
\end{aligned} \tag{8}$$

where

$$\begin{aligned}
\mathcal{L}_t(m_1, m_2, m_3) = & -\text{Li}_2\left(-\frac{m_3}{m_2}\right) - \text{Li}_2\left(-\frac{m_1}{m_2}\right) + \log\left(\frac{m_3}{m_1 + m_2}\right) \log\left(\frac{m_1}{m_2}\right) \\
& - \log\left(\frac{m_3}{m_2}\right) \log\left(\frac{m_2 + m_3}{m_2}\right) - \frac{5}{6}\pi^2 + 2\pi \arctan\left(\sqrt{\frac{m_2(m_1 + m_2 + m_3)}{m_1m_3}}\right).
\end{aligned} \tag{9}$$

The above result is in agreement with [4]. To check it one has to expand in n around $n = 4$ all the factors which depend on n , such as our $C(n)$ (Eq.(6)) and the n -dependent functions of [4].

The functions $\mathcal{G}_i(n, m_1, m_2, m_3)$ $i = 1, 2, 3$ can be easily found in an analogous way as for the pseudothreshold [9] and the solutions read

$$\begin{aligned}
\mathcal{G}_1(n, m_1, m_2, m_3) = & \frac{1}{8m_1m_2} \left\{ - (m_1 + 3m_2) \frac{\partial}{\partial m_1} \mathcal{G}_0(n, m_1, m_2, m_3) + (m_2 - m_3) \frac{\partial}{\partial m_3} \mathcal{G}_0(n, m_1, m_2, m_3) \right. \\
& + \frac{1}{m_1 + m_2 + m_3} \left[((n-3)(2m_1 + m_2 + 2m_3) - m_2) \mathcal{G}_0(n, m_1, m_2, m_3) \right. \\
& + \frac{(n-2)^2}{2(n-3)} \left(\frac{1}{m_1} T(n, m_1^2) T(n, m_2^2) + \frac{m_2}{m_1m_3} T(n, m_1^2) T(n, m_3^2) \right. \\
& \left. \left. + \frac{1}{m_3} T(n, m_2^2) T(n, m_3^2) \right) \right] \left. \right\}
\end{aligned} \tag{10}$$

$$\begin{aligned}
\mathcal{G}_2(n, m_1, m_2, m_3) = & \frac{1}{8m_2^2} \left\{ (3m_1 + m_2) \frac{\partial}{\partial m_1} \mathcal{G}_0(n, m_1, m_2, m_3) + (m_2 + 3m_3) \frac{\partial}{\partial m_3} \mathcal{G}_0(n, m_1, m_2, m_3) \right. \\
& + \frac{1}{m_1 + m_2 + m_3} \left[-((n-3)(6m_1 + 7m_2 + 6m_3) + m_2) \mathcal{G}_0(n, m_1, m_2, m_3) \right.
\end{aligned}$$

$$+\frac{(n-2)^2}{2(n-3)} \left\{ \left(\frac{1}{m_1} T(n, m_1^2) T(n, m_2^2) + \frac{m_2}{m_1 m_3} T(n, m_1^2) T(n, m_3^2) \right. \right. \\ \left. \left. + \frac{1}{m_3} T(n, m_2^2) T(n, m_3^2) \right) \right\} \quad (11)$$

$$\mathcal{G}_3(n, m_1, m_2, m_3) = \\ \frac{1}{8m_2 m_3} \left\{ - (m_1 - m_2) \frac{\partial}{\partial m_1} \mathcal{G}_0(n, m_1, m_2, m_3) - (3m_2 + m_3) \frac{\partial}{\partial m_3} \mathcal{G}_0(n, m_1, m_2, m_3) \right. \\ \left. + \frac{1}{m_1 + m_2 + m_3} \left[((n-3)(2m_1 + m_2 + 2m_3) - m_2) \mathcal{G}_0(n, m_1, m_2, m_3) \right. \right. \\ \left. \left. + \frac{(n-2)^2}{2(n-3)} \left(\frac{1}{m_1} T(n, m_1^2) T(n, m_2^2) + \frac{m_2}{m_1 m_3} T(n, m_1^2) T(n, m_3^2) \right. \right. \right. \\ \left. \left. \left. + \frac{1}{m_3} T(n, m_2^2) T(n, m_3^2) \right) \right] \right\} \quad (12)$$

where $T(n, m^2)$ is defined by

$$T(n, m^2) = \int \frac{d^n k}{(2\pi)^{n-2}} \frac{1}{k^2 + m^2} = \frac{m^{n-2}}{(n-2)(n-4)} C(n) . \quad (13)$$

After expanding around $n = 4$ one gets, besides of Eq.(7),

$$\mathcal{G}_3^{(0)}(m_1, m_2, m_3) = \frac{1}{8(m_1 + m_2 + m_3)^2} \left[2\pi \sqrt{m_1 m_2 m_3 (m_1 + m_2 + m_3)} \right. \\ \left. + m_1^2 \mathcal{L}_t(m_1, m_2, m_3) + m_2^2 \mathcal{L}_t(m_2, m_1, m_3) \right] \\ + \frac{1}{8(m_1 + m_2 + m_3)} \left[m_1 \log \left(\frac{m_1}{m_3} \right) + m_2 \log \left(\frac{m_2}{m_3} \right) \right] \\ - \frac{1}{8} \left[\mathcal{L}_t(m_1, m_2, m_3) + \mathcal{L}_t(m_2, m_1, m_3) - \log^2(m_3) \right. \\ \left. - \log(m_3) \log(m_1) - \log(m_3) \log(m_2) + \log(m_1) \log(m_2) + \log(m_3) + \frac{1}{4} \right] . \quad (14)$$

The other two functions, $\mathcal{G}_1^{(0)}(m_1, m_2, m_3)$ and $\mathcal{G}_2^{(0)}(m_1, m_2, m_3)$, can be easily obtained by a permutation of the masses and we do not report them here. Again the results are in agreement with [4].

3 The expansion at the threshold.

The expansions at the threshold of the master amplitudes can be found with the help of the system of equations obtained in [1] (Eqs.(5,7)). As the threshold is a singular point the expansions consist of the regular parts plus the singular ones with fractional exponents. By inserting the expansion into the system of equations one finds that the fractional powers are fixed and the expansion reads

$$F_\alpha(n, m_1^2, m_2^2, m_3^2, p^2) = \sum_{i=0}^{\infty} \mathcal{H}^{(\alpha, i)}(n, m_1, m_2, m_3) x^i + x^{n-j(\alpha)} \mathcal{H}_s^{(\alpha, 0)}(n, m_1, m_2, m_3) \left(1 + \sum_{i=1}^{\infty} \mathcal{H}_s^{(\alpha, i)}(n, m_1, m_2, m_3) x^i \right), \quad (15)$$

where n is the continuous dimension, $\alpha = 0, 1, 2, 3$; $j(0) = 2$, $j(1) = j(2) = j(3) = 3$ and

$$x = p^2 + (m_1 + m_2 + m_3)^2 \quad (16)$$

is the expansion parameter. Of course in this new notation

$$\mathcal{H}^{(\alpha, 0)}(n, m_1, m_2, m_3) \equiv \mathcal{G}_\alpha(n, m_1, m_2, m_3) \quad (17)$$

is already presented in Eq.(5).

From Eq.(3) it follows that the $\mathcal{H}^{(\alpha, i)}(n, m_1, m_2, m_3)$ and $\mathcal{H}_s^{(\alpha, i)}(n, m_1, m_2, m_3)$ are related, so that having $\mathcal{H}^{(0, i)}(n, m_1, m_2, m_3)$ and $\mathcal{H}_s^{(0, i)}(n, m_1, m_2, m_3)$ one can calculate the others for $\alpha = 1, 2, 3$.

From the discussed system of equations follows also that the following differential equations hold for $\mathcal{H}_s^{(0, 0)}(n, m_1, m_2, m_3)$

$$\begin{aligned} \frac{\partial \mathcal{H}_s^{(0, 0)}(n, m_1, m_2, m_3)}{\partial m_1} &= \left[\frac{n-3}{2m_1} + \frac{5-3n}{2(m_1+m_2+m_3)} \right] \mathcal{H}_s^{(0, 0)}(n, m_1, m_2, m_3) \\ \frac{\partial \mathcal{H}_s^{(0, 0)}(n, m_1, m_2, m_3)}{\partial m_2} &= \left[\frac{n-3}{2m_2} + \frac{5-3n}{2(m_1+m_2+m_3)} \right] \mathcal{H}_s^{(0, 0)}(n, m_1, m_2, m_3) \\ \frac{\partial \mathcal{H}_s^{(0, 0)}(n, m_1, m_2, m_3)}{\partial m_3} &= \left[\frac{n-3}{2m_3} + \frac{5-3n}{2(m_1+m_2+m_3)} \right] \mathcal{H}_s^{(0, 0)}(n, m_1, m_2, m_3). \end{aligned} \quad (18)$$

That means that $\mathcal{H}_s^{(0, 0)}(n, m_1, m_2, m_3)$ can be written as

$$\mathcal{H}_s^{(0,0)}(n, m_1, m_2, m_3) = \mathcal{C}_\mathcal{H}(n) \frac{(m_1 m_2 m_3)^{\frac{n-3}{2}}}{(m_1 + m_2 + m_3)^{\frac{3n-5}{2}}} , \quad (19)$$

where $\mathcal{C}_\mathcal{H}(n)$ is a function of n only, to be discussed later. As usual, the coefficients $\mathcal{H}_s^{(0,i)}(n, m_1, m_2, m_3)$, $i = 1, \dots$ can be found by solving a system of (in this case four) linear equations. We report here for brevity only one of the coefficients

$$\mathcal{H}_s^{(0,1)}(n, m_1, m_2, m_3) = \frac{1}{16(m_1 + m_2 + m_3)} \left[\frac{3(3n-5)}{m_1 + m_2 + m_3} - (n-3) \left(\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} \right) \right] . \quad (20)$$

The singular part of the expansion in Eq.(15) can be compared with the results of [7], as it corresponds, for arbitrary masses and in the language of that reference, to the (p-p) regions contribution completely expanded at the threshold. Our results Eq.(15), Eq.(19) and Eq.(20) agree analytically with those presented in Eq.(5), Eq.(49), Eq.(50) of [7], provided that the value given in Eq.(49) of [7] is taken with a minus sign, in agreement with its equal mass limit, Eq.(51) of [7].

In the regular part of the expansion all the coefficients can be found provided we know $\mathcal{H}^{(\alpha,0)}(n, m_1, m_2, m_3) \equiv \mathcal{G}_\alpha(n, m_1, m_2, m_3)$, for $\alpha = 0, 1, 2, 3$, already given in Eq.(5). This is only partly true as we know $\mathcal{H}^{(\alpha,0)}(n, m_1, m_2, m_3)$ expanded around $n = 4$ up to the constant term only. The next to lowest term in the x -expansion reads

$$\begin{aligned} \mathcal{H}^{(0,1)}(n, m_1, m_2, m_3) = & -\frac{1}{(m_1 + m_2 + m_3)^2} \left[(n-3) \mathcal{G}_0(n, m_1, m_2, m_3) \right. \\ & \left. + m_1^2 \mathcal{G}_1(n, m_1, m_2, m_3) + m_2^2 \mathcal{G}_2(n, m_1, m_2, m_3) + m_3^2 \mathcal{G}_3(n, m_1, m_2, m_3) \right] . \end{aligned} \quad (21)$$

The Eq.(21) allows us to find the expansion of $\mathcal{H}^{(0,1)}(n, m_1, m_2, m_3)$ around $n = 4$ up to the constant term as we know the expansion of $\mathcal{G}_\alpha(n, m_1, m_2, m_3)$ up to the constant terms. It can be written as

$$\mathcal{H}^{(0,1)}(n, m_1, m_2, m_3) = C^2(n) \left[\frac{1}{n-4} \mathcal{H}_{(0,1)}^{(-1)} + \mathcal{H}_{(0,1)}^{(0)} + O(n-4) \right] , \quad (22)$$

with

$$\mathcal{H}_{(0,1)}^{(-1)} = \frac{1}{32} , \quad (23)$$

and

$$\begin{aligned} \mathcal{H}_{(0,1)}^{(0)} = & -\frac{1}{8(m_1 + m_2 + m_3)^4} \left[2\pi \left(m_1^2 + m_1 m_2 + m_2^2 \right) \sqrt{m_1 m_2 m_3 (m_1 + m_2 + m_3)} \right. \\ & \left. + m_1^3 (m_1 + 2m_2) \mathcal{L}_t(m_1, m_2, m_3) + m_2^3 (2m_1 + m_2) \mathcal{L}_t(m_2, m_1, m_3) \right] \\ & -\frac{1}{16(m_1 + m_2 + m_3)^3} \left[-4\pi (m_1 + m_2) \sqrt{m_1 m_2 m_3 (m_1 + m_2 + m_3)} \right. \\ & - 4(m_1 + m_2) \left(m_1^2 \mathcal{L}_t(m_1, m_2, m_3) + m_2^2 \mathcal{L}_t(m_2, m_1, m_3) \right) \\ & + \left(m_1^2 + m_1 m_2 + m_2^2 \right) \left(m_1 \log(m_1^2) + m_2 \log(m_2^2) \right) \\ & \left. - \left(m_1^3 + 2m_1^2 m_2 + 2m_1 m_2^2 + m_2^3 \right) \log(m_3^2) \right] \\ & -\frac{1}{32(m_1 + m_2 + m_3)^2} \left[4m_1^2 \mathcal{L}_t(m_1, m_2, m_3) + 4m_2^2 \mathcal{L}_t(m_2, m_1, m_3) - 2(m_1^2 + m_1 m_2 + m_2^2) \right. \\ & \left. - m_1 (3m_1 + 2m_2) \log(m_1^2) - m_2 (2m_1 + 3m_2) \log(m_2^2) + (3m_1^2 + 4m_1 m_2 + 3m_2^2) \log(m_3^2) \right] \\ & -\frac{m_1 + m_2}{16(m_1 + m_2 + m_3)} + \frac{1}{32} \log(m_3^2) - \frac{5}{128} . \end{aligned} \quad (24)$$

In the equal mass case ($m_1 = m_2 = m_3$) the above result is in agreement, up to terms $O(n-4)$, with [7] , while the analytical result for all masses different was not previously known in the literature.

In order to obtain the first order term in the x -expansion for the other master amplitudes, we choose to proceed with the expansion of $F_0(n, m_1^2, m_2^2, m_3^2, p^2)$. It is necessary to know terms of the order x^2 of $F_0(n, m_1^2, m_2^2, m_3^2, p^2)$ to find terms of the order x of the other master amplitudes using the relations of Eq.(3). For the expression of $\mathcal{H}^{(0,2)}(n, m_1, m_2, m_3)$ we find

$$\mathcal{H}^{(0,2)}(n, m_1, m_2, m_3) = \frac{1}{32(n-4)} \frac{1}{(m_1 + m_2 + m_3)^3}$$

$$\left[(2m_1 + m_2 + m_3)\mathcal{G}_1(n, m_1, m_2, m_3) + (m_1 + 2m_2 + m_3)\mathcal{G}_2(n, m_1, m_2, m_3) \right. \\ \left. + (m_1 + m_2 + 2m_3)\mathcal{G}_3(n, m_1, m_2, m_3) \right] + \dots \quad (25)$$

We do not write here explicitly other terms, which are known, containing up to triple pole in the $(n-4)$ expansion. In fact as one can see later (Eq.(33)) the triple and double pole terms cancel in the expansion. The presence of the explicitly shown term $\sim \frac{1}{n-4}$ is a kind of an obstacle analogous to the one encountered at the pseudothreshold expansion [9]. In principle it requires the knowledge of the expansion of the combination of the master integrals written in square brackets of Eq.(25) up to the term $\sim (n-4)$ included. We could solve that problem in an analogous way as in the pseudothreshold expansion [9], but to show how far we can get just relying on the differential equations we will proceed differently. Defining the above combination

$$\mathcal{G}_c(n, m_1, m_2, m_3) = \left[(2m_1 + m_2 + m_3)\mathcal{G}_1(n, m_1, m_2, m_3) \right. \\ \left. + (m_1 + 2m_2 + m_3)\mathcal{G}_2(n, m_1, m_2, m_3) + (m_1 + m_2 + 2m_3)\mathcal{G}_3(n, m_1, m_2, m_3) \right], \quad (26)$$

we calculate the m_3 derivative of it

$$\frac{\partial \mathcal{G}_c(n, m_1, m_2, m_3)}{\partial m_3} = \\ (n-3)\mathcal{G}_c(n, m_1, m_2, m_3) \left(\frac{3}{2m_3} + \frac{1}{2(m_1 + m_2 + m_3)} - \frac{2}{m_1 + m_2 + 2m_3} \right) \\ + \mathcal{G}_c(n, m_1, m_2, m_3) \left(-\frac{1}{m_3} - \frac{1}{m_1 + m_2 + m_3} + \frac{2}{m_1 + m_2 + 2m_3} \right) \\ + (n-4)\mathcal{G}_1(n, m_1, m_2, m_3) \left(-1 - \frac{2m_1 + m_2}{m_3} + \frac{3m_1 + m_2}{m_1 + m_2 + 2m_3} \right) \\ + (n-4)\mathcal{G}_2(n, m_1, m_2, m_3) \left(-1 - \frac{m_1 + 2m_2}{m_3} + \frac{m_1 + 3m_2}{m_1 + m_2 + 2m_3} \right) \\ + \frac{(n-2)^2}{4m_2^2 m_3^2} T(n, m_2^2) T(n, m_3^2) \left(1 - \frac{m_2}{2m_3} - \frac{m_1}{2(m_1 + m_2 + m_3)} \right) \\ + \frac{(n-2)^2}{4m_1^2 m_3^2} T(n, m_1^2) T(n, m_3^2) \left(1 - \frac{m_1}{2m_3} - \frac{m_2}{2(m_1 + m_2 + m_3)} \right) \\ + \frac{(n-2)^2}{4m_1^2 m_2^2} T(n, m_1^2) T(n, m_2^2) \left(-\frac{m_1 + m_2}{2m_3} + \frac{m_1 + m_2}{2(m_1 + m_2 + m_3)} \right). \quad (27)$$

When expanding around $n = 4$, we find (due to the factor $(n - 4)$ in front of $\mathcal{G}_1(n, m_1, m_2, m_3)$ and $\mathcal{G}_2(n, m_1, m_2, m_3)$) that the term of $\mathcal{G}_c(n, m_1, m_2, m_3)$ proportional to $(n - 4)$, denoted as $G_c^{(1)}(m_1, m_2, m_3)$,

$$\mathcal{G}_c(n, m_1, m_2, m_3) = C^2(n) \left[\cdots + (n - 4) G_c^{(1)}(m_1, m_2, m_3) + \cdots \right], \quad (28)$$

fulfills the following first order differential equation

$$\begin{aligned} \frac{\partial G_c^{(1)}(m_1, m_2, m_3)}{\partial m_3} = & \frac{1}{2} G_c^{(1)}(m_1, m_2, m_3) \left(\frac{1}{m_3} - \frac{1}{m_1 + m_2 + m_3} \right) \\ & + \frac{1}{2} \mathcal{G}_1^{(0)}(m_1, m_2, m_3) \left(\frac{2m_1 + m_2}{m_3} + \frac{m_1}{m_1 + m_2 + m_3} \right) \\ & + \frac{1}{2} \mathcal{G}_2^{(0)}(m_1, m_2, m_3) \left(\frac{2m_2 + m_1}{m_3} + \frac{m_2}{m_1 + m_2 + m_3} \right) \\ & + \frac{1}{2} \mathcal{G}_3^{(0)}(m_1, m_2, m_3) \left(4 + \frac{3(m_1 + m_2)}{m_3} - \frac{m_1 + m_2}{m_1 + m_2 + m_3} \right) \\ & - \frac{1}{384} \log^3(m_1^2) \left(-2 + \frac{2m_1 + m_2}{m_3} - \frac{m_1}{m_1 + m_2 + m_3} \right) \\ & - \frac{1}{384} \log^3(m_2^2) \left(-2 + \frac{m_1 + 2m_2}{m_3} - \frac{m_2}{m_1 + m_2 + m_3} \right) \\ & - \frac{1}{384} \log^3(m_3^2) \left(-4 + \frac{m_1 + m_2}{m_3} + \frac{m_1 + m_2}{m_1 + m_2 + m_3} \right) \\ & - \frac{1}{128} \log(m_1^2) \log(m_2^2) \left(\log(m_1^2) + \log(m_2^2) \right) \left(\frac{m_1 + m_2}{m_3} - \frac{m_1 + m_2}{m_1 + m_2 + m_3} \right) \\ & - \frac{1}{128} \log(m_1^2) \log(m_3^2) \left(\log(m_1^2) + \log(m_3^2) \right) \left(-2 + \frac{m_1}{m_3} + \frac{m_2}{m_1 + m_2 + m_3} \right) \\ & - \frac{1}{128} \log(m_2^2) \log(m_3^2) \left(\log(m_2^2) + \log(m_3^2) \right) \left(-2 + \frac{m_2}{m_3} + \frac{m_1}{m_1 + m_2 + m_3} \right), \quad (29) \end{aligned}$$

instead of being a part of a system of three differential equations, as in the case of arbitrary n .

We can find a solution of that equation in a relatively simple way, which consists mainly in integrating by parts, so we do not report here details of the derivation. The solution reads

$$G_c^{(1)}(m_1, m_2, m_3) = \frac{(m_2 - m_3)(2m_1 + m_2 + m_3)}{4(m_1 + m_2 + m_3)} \mathcal{L}_t(m_1, m_2, m_3)$$

$$\begin{aligned}
& + \frac{(m_1 - m_3)(m_1 + 2m_2 + m_3)}{4(m_1 + m_2 + m_3)} \mathcal{L}_t(m_2, m_1, m_3) \\
& + \frac{1}{192} \left[\log^3(m_1^2) (2m_1 + m_2 + m_3) + \log^3(m_2^2) (m_1 + 2m_2 + m_3) \right. \\
& \quad \left. + \log^3(m_3^2) (m_1 + m_2 + 2m_3) \right] \\
& + \frac{1}{64} \log(m_1^2) \log(m_2^2) (\log(m_1^2) + \log(m_2^2)) (m_1 + m_2) \\
& + \frac{1}{64} \log(m_1^2) \log(m_3^2) (\log(m_1^2) + \log(m_3^2)) (m_1 + m_3) \\
& + \frac{1}{64} \log(m_2^2) \log(m_3^2) (\log(m_2^2) + \log(m_3^2)) (m_2 + m_3) \\
& - \frac{1}{32} \left[m_3 (\log(m_1^2) + \log(m_2^2))^2 + m_2 (\log(m_1^2) + \log(m_3^2))^2 + m_1 (\log(m_2^2) + \log(m_3^2))^2 \right] \\
& + \frac{1}{8} \left[\log(m_1^2) (-3m_1 + m_2 + m_3) + \log(m_2^2) (m_1 - 3m_2 + m_3) \right. \\
& \quad \left. + \log(m_3^2) (m_1 + m_2 - 3m_3) \right] + \frac{11}{16} (m_1 + m_2 + m_3) \\
& + \sqrt{\frac{m_1 m_2 m_3}{m_1 + m_2 + m_3}} \left\{ \pi \left[\frac{3}{4} (\log(m_1^2) + \log(m_2^2) + \log(m_3^2)) + \frac{1}{2} \log(m_1 + m_2 + m_3) \right. \right. \\
& \quad \left. \left. - \log(m_1 + m_2) - \log(m_1 + m_3) - \log(m_2 + m_3) \right] + \mathcal{I}_3(m_1, m_2, m_3) - K \right\}, \quad (30)
\end{aligned}$$

where the unknown function of m_1 and m_2 , which remains after integration, is reduced to the single constant K using the symmetry of $G_c^{(1)}(m_1, m_2, m_3)$ under the interchange of all the masses. The actual value of K will be fixed later. $\mathcal{L}_t(m_1, m_2, m_3)$ is defined in Eq.(9) and the only non trivial integral left, $\mathcal{I}_3(m_1, m_2, m_3)$, is

$$\begin{aligned}
\mathcal{I}_3(m_1, m_2, m_3) &= \tilde{\mathcal{I}}_3(m_1, m_2, m_3) + \tilde{\mathcal{I}}_3(m_1, m_1, m_2) - \tilde{\mathcal{I}}_3(m_2, m_1, m_1), \\
\tilde{\mathcal{I}}_3(m_1, m_2, m_3) &= \sqrt{m_1 m_2} \int dm_3 \frac{1}{\sqrt{m_3(m_1 + m_2 + m_3)}} \left[\frac{\log\left(\frac{m_3}{m_1}\right)}{m_3 + m_1} + \frac{\log\left(\frac{m_3}{m_2}\right)}{m_3 + m_2} \right] = \\
& i \left\{ \log\left(\frac{m_1 + m_2}{4m_1}\right) [\log(t - t_1) - \log(t - t_2)] + \log\left(\frac{m_1 + m_2}{4m_2}\right) [\log(t + t_2) - \log(t + t_1)] \right. \\
& \quad + \log(t - t_1) [2 \log(1 - t_1) - \log(t_1)] - \log(t - t_2) [2 \log(1 - t_2) - \log(t_2)] \\
& \quad - \log(t + t_1) [2 \log(1 + t_1) - \log(-t_1)] + \log(t + t_2) [2 \log(1 + t_2) - \log(-t_2)] \\
& \quad \left. - 2 \operatorname{Li}_2\left(\frac{t - t_1}{1 - t_1}\right) + 2 \operatorname{Li}_2\left(\frac{t - t_2}{1 - t_2}\right) + \operatorname{Li}_2\left(-\frac{t - t_1}{t_1}\right) - \operatorname{Li}_2\left(-\frac{t - t_2}{t_2}\right) \right\}
\end{aligned}$$

$$+2 \operatorname{Li}_2 \left(\frac{t+t_1}{1+t_1} \right) - 2 \operatorname{Li}_2 \left(\frac{t+t_2}{1+t_2} \right) - \operatorname{Li}_2 \left(\frac{t+t_1}{t_1} \right) + \operatorname{Li}_2 \left(\frac{t+t_2}{t_2} \right) \Big\} , \quad (31)$$

where t and $t_{1,2}$ are defined as

$$\begin{aligned} t &= \frac{\sqrt{m_1+m_2+m_3} - \sqrt{m_3}}{\sqrt{m_1+m_2+m_3} + \sqrt{m_3}} , \\ t_{1,2} &= \frac{m_2 - m_1 \pm 2i\sqrt{m_1 m_2}}{m_1 + m_2} . \end{aligned} \quad (32)$$

Having $G_c^{(1)}(m_1, m_2, m_3)$ we can find the $n = 4$ expansion of $\mathcal{H}^{(0,2)}(n, m_1, m_2, m_3)$ up to the constant term. It reads

$$\mathcal{H}^{(0,2)}(n, m_1, m_2, m_3) = C^2(n) \left[\frac{1}{(n-4)} \mathcal{H}_{(0,2)}^{(-1)} + \mathcal{H}_{(0,2)}^{(0)} + O(n-4) \right] , \quad (33)$$

where

$$\mathcal{H}_{(0,2)}^{(-1)} = \frac{\pi}{32} \frac{\sqrt{m_1 m_2 m_3 (m_1 + m_2 + m_3)}}{(m_1 + m_2 + m_3)^4} , \quad (34)$$

and

$$\begin{aligned} \mathcal{H}_{(0,2)}^{(0)} = & -\frac{1}{8(m_1+m_2+m_3)^6} \left[m_1^3(m_1+2m_2)\mathcal{L}_t(m_1, m_2, m_3) + m_2^3(2m_1+m_2)\mathcal{L}_t(m_2, m_1, m_3) \right] \\ & + \frac{1}{16(m_1+m_2+m_3)^5} \left[4(m_1+m_2) \left(m_1^2\mathcal{L}_t(m_1, m_2, m_3) + m_2^2\mathcal{L}_t(m_2, m_1, m_3) \right) \right. \\ & \quad - m_1 \log(m_1^2) \left(m_1^2 + m_1 m_2 + m_2^2 \right) - m_2 \log(m_2^2) \left(m_1^2 + m_1 m_2 + m_2^2 \right) \\ & \quad \left. + \log(m_3^2) \left(m_1^3 + 2m_1^2 m_2 + 2m_1 m_2^2 + m_2^3 \right) \right] \\ & + \frac{1}{32(m_1+m_2+m_3)^4} \left[-4m_1^2\mathcal{L}_t(m_1, m_2, m_3) - 4m_2^2\mathcal{L}_t(m_2, m_1, m_3) \right. \\ & \quad + m_1 \log(m_1^2) (3m_1+2m_2) + m_2 \log(m_2^2) (2m_1+3m_2) \\ & \quad \left. - \log(m_3^2) \left(3m_1^2 + 4m_1 m_2 + 3m_2^2 \right) + 2(m_1^2 + m_1 m_2 + m_2^2) \right] \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{32(m_1 + m_2 + m_3)^3} \left[-m_1 \log(m_1^2) - m_2 \log(m_2^2) + (m_1 + m_2) \log(m_3^2) - 2(m_1 + m_2) \right] \\
& + \frac{1}{64(m_1 + m_2 + m_3)^2} - \frac{\pi \sqrt{m_1 m_2 m_3 (m_1 + m_2 + m_3)}}{4(m_1 + m_2 + m_3)^6} (m_1^2 + m_1 m_2 + m_2^2) \\
& + \frac{\pi \sqrt{m_1 m_2 m_3 (m_1 + m_2 + m_3)}}{4(m_1 + m_2 + m_3)^5} (m_1 + m_2) \\
& + \frac{\sqrt{m_1 m_2 m_3 (m_1 + m_2 + m_3)}}{64(m_1 + m_2 + m_3)^4} \left\{ \pi \left[\frac{3}{2} (\log(m_1^2) + \log(m_2^2) + \log(m_3^2)) \right. \right. \\
& \left. \left. - 2(\log(m_1 + m_2) + \log(m_1 + m_3) + \log(m_2 + m_3)) + \log(m_1 + m_2 + m_3) - 3 \right] \right. \\
& \left. + 2\mathcal{I}_3(m_1, m_2, m_3) - 2K \right\}. \tag{35}
\end{aligned}$$

As we know only the expansion of $\mathcal{H}^{(0,2)}(n, m_1, m_2, m_3)$ around $n = 4$ and not its exact form for arbitrary n , to proceed, we write also the expansion in $(n - 4)$ of the complete first master amplitude $F_0(n, m_1^2, m_2^2, m_3^2, p^2)$. The singular and regular part are well distinct for continuous arbitrary n ; in the expansion around $n = 4$, the singular part gives terms proportional to $\log(x)$ plus other terms, without $\log(x)$, which mix with the terms coming from the regular part. Expanding Eq.(15) in $(n - 4)$ we find

$$\begin{aligned}
F_0(n, m_1^2, m_2^2, m_3^2, p^2) = C^2(n) \left\{ \right. \\
& \frac{1}{(n - 4)^2} \mathcal{G}_0^{(-2)}(m_1, m_2, m_3) + \frac{1}{(n - 4)} \mathcal{G}_0^{(-1)}(m_1, m_2, m_3) + \mathcal{G}_0^{(0)}(m_1, m_2, m_3) \\
& + x \left[\frac{1}{(n - 4)} \mathcal{H}_{(0,1)}^{(-1)} + \mathcal{H}_{(0,1)}^{(0)} \right] \\
& + x^2 \left[\mathcal{H}_{(0,2)}^{(0)} - \mathcal{H}_{(0,2)}^{(-1)} \left(\log(x) + \frac{1}{2} \log \left(\frac{m_1 m_2 m_3}{(m_1 + m_2 + m_3)^3} \right) \right) \right. \\
& \left. + b \frac{\sqrt{m_1 m_2 m_3 (m_1 + m_2 + m_3)}}{(m_1 + m_2 + m_3)^4} \right] + O(n - 4, x^3) \left. \right\}, \tag{36}
\end{aligned}$$

where the constant b comes from the expansion of $\mathcal{C}_\mathcal{H}(n)$ around $n = 4$

$$\mathcal{C}_\mathcal{H}(n) = C^2(n) \left[-\frac{1}{(n - 4)} \frac{\pi}{32} + b + O(n - 4) \right]. \tag{37}$$

The requirement of the disappearance of the pole term in the coefficient of x^2 in Eq.(36) and the knowledge of the pole term of $\mathcal{H}^{(0,2)}(n, m_1, m_2, m_3)$ fix the pole term in $\mathcal{C}_{\mathcal{H}}(n)$. That requirement fix also the absence of higher pole terms in $\mathcal{C}_{\mathcal{H}}(n)$. In the expansion Eq.(36) there are two still unknown constants: K in the expression of $\mathcal{H}_{(0,2)}^{(0)}$ and b from $\mathcal{C}_{\mathcal{H}}(n)$. They cannot be fixed separately due to the $n = 4$ expansion, but only in the occurring combination $(b - K/32)$. As we are interested only in the $n = 4$ expansion of the master integrals, the knowledge of the combination of the constants appearing in Eq.(36) is enough to fix all the higher order terms in the threshold expansion. To fix the combination of the constants it is sufficient to know the term $\sim x^2$ of $F_0(n, m_1^2, m_2^2, m_3^2, p^2)$ for fixed values of the masses. Due to the factor in front of b and K , $\sqrt{m_1 m_2 m_3}$, all masses have to be different from zero and so the simplest choice is the equal mass case ($m_1 = m_2 = m_3 = m$). As the analytical result for equal mass case is available [7], we limited ourselves to its numerical check, which can be performed by using the dispersion relation representation of the master integral $F_0(n = 4, m^2, m^2, m^2, p^2)$ [9]. Its second derivative reads

$$\frac{\partial^2 F_0(n = 4, m^2, m^2, m^2, p^2)}{\partial (p^2)^2} = \int_{9m^2}^{\infty} du \frac{1}{8(u + p^2)^3} E_0(u, m) , \quad (38)$$

where

$$E_0(u, m) = \frac{1}{u} \int_{4m^2}^{(\sqrt{u}-m)^2} db \frac{R(u, b, m^2) R(b, m^2, m^2)}{b} . \quad (39)$$

At $p^2 = -9m^2$ the integral Eq.(38) is logarithmically divergent (in agreement with Eq.(36)), but the divergent part can be easily extracted by integrating by parts, providing us with the two needed leading terms in the threshold expansion of the integral

$$\begin{aligned} \frac{\partial^2 F_0(n = 4, m^2, m^2, m^2, p^2)}{\partial (p^2)^2} &= -\frac{\pi\sqrt{3}}{6^4 m^2} \log\left(\frac{x_e}{m^2}\right) \\ &- \frac{1}{16} \int_{9m^2}^{\infty} du \log\left(\frac{u - 9m^2}{m^2}\right) \frac{\partial^3 E_0(u, m)}{\partial u^3} + O(x_e) \end{aligned} \quad (40)$$

$$\begin{aligned} &= -\frac{\pi\sqrt{3}}{6^4 m^2} \log\left(\frac{x_e}{m^2}\right) \\ &- \frac{1}{864 m^2} + \frac{\pi\sqrt{3}}{m^2} \left(-\frac{1}{972} + \frac{\log(2)}{432} + \frac{\log(3)}{648} \right) - \frac{5\sqrt{3}}{1296 m^2} \text{Cl}_2\left(\frac{\pi}{3}\right) + O(x_e) , \end{aligned} \quad (41)$$

where $x_e = p^2 + 9m^2$ corresponds to the equal mass limit $m_1 = m_2 = m_3 = m$ of x in Eq.(16) and the explicit value of the integral is taken from [7]. The analytical result of [7], in Eq.(41), agrees with the numerical value of the integral in Eq.(40), so we did not reevaluate analytically the integral.

For extracting the combination of the constants $(b - K/32)$, we consider Eq.(36) in the equal mass limit $m_1 = m_2 = m_3 = m$, and we compare its second p^2 derivative with Eq.(41), obtaining

$$b - \frac{K}{32} = \pi \left(-\frac{1}{32} + \frac{5}{32} \log(2) \right) + \frac{1}{8} \text{Cl}_2 \left(\frac{\pi}{2} \right), \quad (42)$$

to be used in Eq.(36) and in all the other $n = 4$ expansions of the master integrals. For completeness we report Eq.(36) in the equal mass limit $m_1 = m_2 = m_3 = m$ in our notations

$$\begin{aligned} F_0(n, m^2, m^2, m^2, p^2) = & C^2(n) \left\{ -\frac{3m^2}{8(n-4)^2} + \frac{3m^2}{32(n-4)} (3 - 4 \log(m^2)) \right. \\ & - \frac{m^2 \pi \sqrt{3}}{6} + \frac{3m^2}{128} (15 + 12 \log(m^2) - 8 \log^2(m^2)) \\ & + x_e \left[\frac{1}{32(n-4)} + \frac{\pi \sqrt{3}}{108} - \frac{23}{384} + \frac{1}{32} \log(m^2) \right] \\ & + x_e^2 \frac{\sqrt{3}}{648m^2} \left[\pi \left(-\frac{1}{4} \log \left(\frac{x_e}{m^2} \right) + \frac{3}{4} \log(2) + \frac{1}{2} \log(3) + \frac{1}{24} \right) - \frac{5}{4} \text{Cl}_2 \left(\frac{\pi}{3} \right) - \frac{\sqrt{3}}{8} \right] \\ & \left. + O(n-4, x_e^3) \right\}. \end{aligned} \quad (43)$$

The higher order terms in the expansion at threshold can be easily obtained algebraically, but they are of interest only in case one wants to calculate the master amplitudes using the threshold expansion [7], which is not our purpose here. The expansion of all the master integrals up to terms $\sim x$ is however necessary for the numerical solution of the system of equations obtained in [1].

The threshold expansion of the remaining master integrals can be obtained using Eq.(3). One gets

$$F_3(n, m_1^2, m_2^2, m_3^2, p^2) = C^2(n) \left[\frac{1}{(n-4)^2} \mathcal{G}_3^{(-2)}(m_1, m_2, m_3) + \frac{1}{(n-4)} \mathcal{G}_3^{(-1)}(m_1, m_2, m_3) \right]$$

$$+\mathcal{G}_3^{(0)}(m_1, m_2, m_3) + x \mathcal{H}_{(3,1)}^{(0)} + O(n-4, x^2) \Big], \quad (44)$$

where $\mathcal{G}_j^{(i)}(m_1, m_2, m_3)$ are defined in Eq.(7) and Eq.(14) and

$$\begin{aligned} \mathcal{H}_{(3,1)}^{(0)} = & -\pi \frac{\sqrt{m_1 m_2 m_3 (m_1 + m_2 + m_3)}}{4m_3 (m_1 + m_2 + m_3)^4} (m_1 + m_2) \\ & + \frac{\sqrt{m_1 m_2 m_3 (m_1 + m_2 + m_3)}}{8m_3 (m_1 + m_2 + m_3)^3} \left[\pi \left(\frac{1}{2} \log \left(\frac{x}{(m_1 + m_2 + m_3)^2} \right) + 1 + \frac{1}{2} \log \left(\frac{m_2 + m_3}{m_1} \right) \right. \right. \\ & + \frac{1}{2} \log \left(\frac{m_1 + m_3}{m_2} \right) + \frac{1}{2} \log \left(\frac{m_1 + m_2}{m_3} \right) \Big) - 16 \left(b - \frac{K}{32} \right) - \frac{1}{2} \mathcal{I}_3(m_1, m_2, m_3) \Big] \\ & + \frac{1}{8(m_1 + m_2 + m_3)^4} \left[m_1^2 \mathcal{L}_t(m_1, m_2, m_3) + m_2^2 \mathcal{L}_t(m_2, m_1, m_3) \right] \\ & + \frac{1}{16(m_1 + m_2 + m_3)^3} \left[m_1 \log(m_1^2) + m_2 \log(m_2^2) - (m_1 + m_2) \log(m_3^2) \right] \\ & - \frac{1}{16(m_1 + m_2 + m_3)^2} . \end{aligned} \quad (45)$$

The expansion of $F_1(n, m_1^2, m_2^2, m_3^2, p^2)$ and $F_2(n, m_1^2, m_2^2, m_3^2, p^2)$ can be obtained by a permutation of the masses.

4 Summary.

In this paper we have presented the expansion of the 2-loop sunrise selfmass master amplitudes at the threshold $p^2 = -(m_1 + m_2 + m_3)^2$. We define the expansion in Eq.(15); the values of the amplitudes at the threshold are given in Eq.(5), Eq.(7), Eq.(8) and Eq.(14). The first order terms in the threshold expansion of the master amplitudes at $n = 4$ are presented in Eq.(22) and Eq.(44), while only for the first master amplitude $F_0(n = 4, m_1^2, m_2^2, m_3^2, p^2 = -(m_1 + m_2 + m_3)^2)$ the second order term in the threshold expansion is presented in Eq.(36). The higher order terms, which are not given explicitly here, can be easily found by solving recursively, at each order, a system of four algebraic linear equations. As said already in [9], the expansion at the pseudothreshold cannot be simply deduced from the known expansion at the threshold and vice versa, even if at first sight they seem to be connected by the change of sign $m_3 \rightarrow -m_3$. In fact the analytic properties of the amplitudes are different at the two points: at the pseudothreshold the

sunrise amplitudes are regular, so that the solution of the system of equations [1] can be expanded as a single power series, while at the threshold the sunrise amplitudes develop a branch point and its expansion is indeed the sum of two series Eq.(15), [10].

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